

# Independent sets in chain cacti

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## Abstract

In this paper chain cacti are considered. First, for two specific classes of chain cacti (orto-chains and meta-chains of cycles with  $h$  vertices) the recurrence relation for independence polynomial is derived. That recurrence relation is then used in deriving explicit expressions for independence number and number of maximum independent sets for such chains. Also, the recurrence relation for total number of independent sets for such graphs is derived. Finally, the proof is provided that orto-chains and meta-chains are the only extremal chain cacti with respect to total number of independent sets (orto-chains minimal and meta-chains maximal).

*Keywords:*

Cactus graph, Chain cactus graph, Independence set, Independence polynomial

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## 1. Introduction

The notion of cactus graph first appeared in scientific literature in 1950's. Then such graphs were called Husimi trees after the author of the paper which motivated their introduction ([11]), and which was about cluster integrals in statistical mechanics. The same graphs and their generalizations also served as simplified models of real lattices ([12]), they were useful in the theory of electrical and communication networks ([14]) and in chemistry ([15]). Enumerative aspects of such graphs were also studied in various papers ([8],[10]), and finally summarized in classical monograph on graph enumeration by Harary and Palmer ([9]).

Later, these graphs were named cactus graphs in mathematical literature. The interest for such graphs has arisen again recently, since it was noted that

some NP-hard problems can be solved in polynomial time for that class of graphs ([15]). Chain cacti were also studied from aspect of matchings ([4], [5], [6], [7]). The class of graphs very similar to cactus graphs, i.e. block-cactus graphs, was studied recently too ([16]). This paper is motivated by the paper of T. Došlić and F. Måløy ([2]) in which they presented recurrence relations and/or explicit formulas for various invariants related with matchings and independent sets for two specific kind of chain hexagonal cacti. Also, they provided proof that those two kinds of chain hexagonal cacti are extremal among all chain hexagonal cacti with respect to total number of matchings in a graph and with respect to total number of independent sets in graphs. In the same paper they proposed generalizing those results for general chain cacti. The proposed generalization for matchings has been presented in [1]. In this paper the generalization of results by T. Došlić and F. Måløy for independent sets is presented, and therefore this paper nicely supplements [1] in generalizing [2].

The present paper is organized as follows. In the section 'Preliminaries' we introduce some basic notation, the notions about independence and the classes of graphs with which we deal throughout the paper. Third section 'Main results' is divided in three parts. The first part of main results is about ortho-chains, which is special class of chain cacti for which independence polynomials and some results about size and number of maximum independent sets are provided. The second part of main results is about meta-chains, another special class of chain graphs for which the same results as for ortho-chains are provided. Finally, the third part of main results provides the proof of extremality of ortho-chains and meta-chains with respect to total number of independent sets.

## 2. Preliminaries

All graphs considered in this paper will be finite and simple. For a graph  $G$  we denote the set of its vertices with  $V(G)$  (or just  $V$ ) and the set of its edges with  $E = E(G)$  (or just  $E$ ). Subgraph of  $G$  induced by set of vertices  $V' \subseteq V$  will be denoted with  $G[V']$ . For a vertex  $v$  (or set of vertices  $V' \subseteq V$ ) of  $G$  we will denote with  $G - v$  (or with  $G - V'$ ) subgraph of  $G$  induced by  $V \setminus \{v\}$  (or by  $V \setminus V'$ ). For an edge  $e$  (or set of edges  $E' \subseteq E$ ) of  $G$  we will denote with  $G - e$  (or with  $G - E'$ ) subgraph of  $G$  obtained by deleting edge  $e$  (or by deleting set of edges  $E \setminus E'$ ). For a vertex  $v$  of graph  $G$  we will denote with  $N[v]$  set of vertices consisting of  $v$  and all vertices of  $G$  adjacent

to  $V$ . We say that vertex  $v$  of connected graph  $G$  is cut vertex if  $G - v$  is disconnected.

We say that set of vertices  $S \subseteq V$  of graph  $G$  is *independent set* if no two vertices of  $S$  are adjacent in  $G$ . *Size* of an independent set  $S$  is number of vertices contained in  $S$ . An independent set of the largest possible size is called *maximum independent set*. The size of maximum independent set of graph  $G$  is called *independence number* (or the *stability number*) of graph  $G$  and is denoted with  $\alpha(G)$ . The *independence polynomial* of graph  $G$  is defined with

$$i(G; x) = \sum_{k=0}^{\alpha(G)} \Psi_k(G) x^k$$

where  $x$  is a formal variable and  $\Psi_k(G)$  denotes number of independent sets in graph  $G$  of size  $k$ . Obviously,  $\Psi_0(G) = 1$  and  $\Psi_1(G) = |V|$ . Setting  $x = 1$  in  $i(G; x)$  we obtain number of all independent sets in graph  $G$  and denote it with  $\Psi(G)$ . For the notation simplicity sake, we will often write  $i(G)$  instead of  $i(G; x)$  where it doesn't lead to confusion. The following properties of independence polynomials are well known.

**Theorem 1.** *Let  $G$  be a graph and  $v$  its vertex. Then*

$$i(G; x) = i(G - u; x) + x \cdot i(G - N[u]; x).$$

**Theorem 2.** *Let  $G$  be graph consisting of components  $G_1, G_2, \dots, G_k$ . Then*

$$i(G; x) = i(G_1; x) \cdot i(G_2; x) \cdot \dots \cdot i(G_k; x).$$

It is easily verified that for path  $P_n$  and cycle  $C_n$  holds

$$i(P_n; x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-k}{k} x^k,$$

$$i(C_n; x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^k.$$

A *cactus graph* is a connected graph in which each edge is contained in at most one cycle, which means that each block of a cactus graph is an edge or a cycle. A *chain cactus* is cactus in which each block is a cycle which contains at most two cut vertices and each cut vertex is contained in exactly

two cycles. The *length* of chain cactus is number of cycles it consists of. We say that a cycle is *h-cycle* if it contains *h* vertices. An *h-cactus* is a cactus graph in which every block is *h-cycle*. A *chain h-cactus* is cactus which is *h-cactus* and chain cactus.

Let  $A_n$  be a chain *h-cactus*. Cycles of  $A_n$  are denoted with  $C^{(1)}, \dots, C^{(n)}$  by order on chain, vertices of cycle  $C^{(i)}$  are denoted with  $v_k^{(i)}$  by order so that  $v_h^{(i)}$  is cut vertex contained in  $C^{(i-1)}$  too. Vertices of  $C^{(1)}$  are denoted by order so that  $v_1^{(1)}$  is cut vertex. Cut vertices therefore have two labels, but that will not lead to confusion. (Note that completely analogous notation can be introduced to general chain cacti. One only needs to introduce  $h_i$  as number of vertices on  $C^{(i)}$ ).

Cycles  $C^{(1)}$  and  $C^{(n)}$  of  $A_n$  are called *end cycles*, otherwise cycles of  $A_n$  are called *internal cycles*. Note that internal cycles contain exactly two cut vertices, while end cycles contain exactly one cut vertex. We say that an internal cycle  $C^{(j)}$  of  $A_n$  is in *k-position* if its two cut vertices are on distance *k*, i.e. if  $v_h^{(j+1)} = v_k^{(j)}$ . For 1-position and 2-position we introduce names *ortho-* and *meta-position*. If all internal cycles of chain *h-cactus* are in ortho-position, then such cactus is called *ortho-chain* and is denoted with  $O_n$ . If all internal cycles of chain *h-cactus* are in meta-position, then such cactus is called *meta-chain* and is denoted with  $M_n$ .

This notation in chain *h-cacti* (applied on  $O_n$  and  $M_n$ ) is illustrated with Figure 1 and Figure 2.

Figure 1: Notation in orto-chain.

With  $A_{n-j}$  we will denote subcactus of  $A_n$  induced by cycles  $C^{(1)}, \dots, C^{(n-j)}$ . Graph with only one vertex is considered to be chain cactus of length 0 and is denoted with  $A_0$ . Note that there is only one  $A_0$ ,  $A_1$ ,  $A_2$  (these cacti are considered to be both ortho- and meta-).

Figure 2: Notation in meta-chain.

### 3. Main results

For the independence polynomials of short chains holds:

$$\begin{aligned} i(O_0) &= i(A_0) = i(M_0) = 1 + x, \\ i(O_1) &= i(A_1) = i(M_1) = i(C_h) = x \cdot i(P_{h-3}) + i(P_{h-1}), \\ i(O_2) &= i(A_2) = i(M_2) = x \cdot i(P_{h-3})^2 + i(P_{h-1})^2. \end{aligned}$$

These results follow easily from Theorems 1 and 2. Now we want to provide formulae for longer chains, specifically for longer ortho-chains and meta-chains.

#### *Ortho-chains*

The recurrence relation for independence polynomials of longer ortho-chains is given by the following theorem.

**Theorem 3.** *The independence polynomials of  $O_n$ , for  $n \geq 3$ , satisfy*

$$i(O_n) = x \cdot i(P_{h-3})^2 \cdot i(O_{n-2}) + i(P_{h-2}) \cdot i(O_{n-1}).$$

**Proof.** For the independence polynomial of  $O_n$  ( $n \geq 3$ ) holds

$$i(O_n) = x \cdot i(P_{h-3})^2 \cdot i(O_{n-2} - v_1^{(n-2)}) + i(P_{h-1}) \cdot i(O_{n-1} - v_1^{(n-1)}). \quad (1)$$

Similarly, for the independence polynomial of  $O_n - v_1^{(n)}$  holds

$$i(O_n - v_1^{(n)}) = x \cdot i(P_{h-3})^2 \cdot i(O_{n-2} - v_1^{(n-2)}) + i(P_{h-2}) \cdot i(O_{n-1} - v_1^{(n-1)}). \quad (2)$$

Now, since right-hand side of (1) is a linear combination of expressions satisfying (2), that means that it also satisfies (2). Since right-hand side and

left-hand side of (1) are equal, that implies that  $i(O_n)$  satisfies (2) too, and that proves the theorem. ■

By setting  $x = 1$  to the recurrence relation from Theorem 3 we can obtain the recurrence relation for  $\Psi(O_n)$  in which coefficients would be total number of independent sets in different paths. To be precise, we obtain

$$\Psi(O_n) = \Psi(P_{h-3})^2 \cdot \Psi(O_{n-2}) + \Psi(P_{h-2}) \cdot \Psi(O_{n-1}).$$

Given that

$$\Psi(P_n) = \frac{3F_n + L_n}{2}$$

where  $F_n$  is fibonacci and  $L_n$  lucas number, we obtain

$$\Psi(O_n) = \left( \frac{3F_{h-3} + L_{h-3}}{2} \right)^2 \cdot \Psi(O_{n-2}) + \frac{3F_{h-2} + L_{h-2}}{2} \cdot \Psi(O_{n-1}).$$

Therefore, for a specific  $n$  and  $h$  we could calculate exact  $\Psi(O_n)$  from that recurrence relation. Now, we proceed to maximum independent set. We will establish size and number of such sets for ortho-chains, i.e. independence number  $\alpha(O_n)$  and number of maximum independent sets  $\Psi_{\alpha(O_n)}(O_n)$ .

**Theorem 4.** *The independence number of  $O_n$ , for  $n \geq 1$ , is*

$$\alpha(O_n) = \begin{cases} \frac{nh}{2} - \lfloor \frac{n-1}{2} \rfloor, & \text{for } h \text{ even,} \\ \frac{n(h-1)}{2}, & \text{for } h \text{ odd.} \end{cases}$$

**Proof.** For the sake of notation simplicity, let us denote  $\alpha_n = \alpha(O_n)$  and  $p_n = \deg(i(P_n))$ . In the case of even  $h$ , from independence polynomials we obtain

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= \max \{1 + p_{h-3}, p_{h-1}\} = \frac{h}{2}, \\ \alpha_2 &= \max \{1 + 2 \cdot p_{h-3}, 2 \cdot p_{h-1}\} = h. \end{aligned}$$

From recurrence relation of Theorem 3 we obtain

$$\alpha_n = \max \left\{ h - 1 + \alpha_{n-2}, \quad \frac{h-2}{2} + \alpha_{n-1} \right\}.$$

The proof is now by induction on  $n$ . The proof for odd  $h$  is analogous. ■

**Theorem 5.** *The number of maximum independent sets in  $O_n$ , for  $n \geq 2$ , is*

$$\Psi_{\alpha(O_n)}(O_n) = \begin{cases} 1, & \text{for } h \text{ even and } n = 2k, \\ 2 + \frac{kh}{2}, & \text{for } h \text{ even and } n = 2k + 1, \\ \left(\frac{h+1}{2}\right)^2, & \text{for } h \text{ odd.} \end{cases}$$

**Proof.** Let us first prove the result for even  $h$ . For the degree of independence polynomials holds

$$\deg(i(P_{h-3})) = \deg(i(P_{h-2})) = \frac{h-2}{2},$$

with leading coefficient of  $i(P_{h-3})$  being 1. First, we want to establish  $\Psi_{\alpha(O_{2k})}(O_{2k})$  which is a leading coefficient in  $i(O_{2k})$ . Let us recall that by Theorem 3 polynomial  $i(O_{2k})$  satisfies

$$i(O_{2k}) = x \cdot i(P_{h-3})^2 \cdot i(O_{2k-2}) + i(P_{h-2}) \cdot i(O_{2k-1}).$$

We know by Theorem 4 that

$$\deg(i(O_{2k})) = kh - k + 1$$

It is easily verified that

$$\begin{aligned} \deg(x \cdot i(P_{h-3})^2 \cdot i(O_{2k-2})) &= hk - k + 1, \\ \deg(i(P_{h-2}) \cdot i(O_{2k-1})) &= kh - k \end{aligned}$$

These degrees imply  $\Psi_{\alpha(O_{2k})}(O_{2k}) = \Psi_{\alpha(O_{2k-2})}(O_{2k-2})$  (since leading coefficient in  $i(P_{h-3})$  equals 1). Now, from  $\Psi_{\alpha(O_2)}(O_2) = 1$  follows  $\Psi_{\alpha(O_{2k})}(O_{2k}) = 1$ .

Similar reasoning yields

$$\Psi_{\alpha(O_{2k+1})}(O_{2k+1}) = \Psi_{\alpha(O_{2k-1})}(O_{2k-1}) + \frac{1}{2}h$$

which together with  $\Psi_{\alpha(O_1)}(O_1) = 2$  implies the result.

In the case of odd  $h$  we obtain

$$\Psi_{\alpha(O_n)}(O_n) = \Psi_{\alpha(O_{n-1})}(O_{n-1}),$$

and then the result follows from  $\Psi_{\alpha(O_2)}(O_2) = \left(\frac{h+1}{2}\right)^2$ . ■

*Meta-chains*

By similar reasoning one can obtain analogous results for meta-chains.

**Theorem 6.** *The independence polynomials of  $M_n$ , for  $n \geq 3$ , satisfy*

$$i(M_n) = \alpha \cdot i(M_{n-1}) - x^2 \cdot \beta \cdot i(M_{n-2})$$

where

$$\begin{aligned}\alpha &= x^2 \cdot i(P_{h-5}) + x \cdot (i(P_{h-4}) + 2 \cdot i(P_{h-5})) + i(P_{h-4}), \\ \beta &= x \cdot i(P_{h-5})^2 + i(P_{h-5})^2 + i(P_{h-4}) \cdot i(P_{h-5}) - i(P_{h-4}) \cdot i(P_{h-6}).\end{aligned}$$

**Proof.** For the sake of notation simplicity let

$$\begin{aligned}H_n &= M_n - v_2^{(n)}, \\ K_n &= M_n - \{v_1^{(n)}, v_2^{(n)}, v_3^{(n)}\}, \\ p_n &= i(P_n).\end{aligned}$$

Therefore, we have

$$\begin{aligned}i(M_n) &= i(H_n) + x \cdot i(K_n), \\ i(H_n) &= i(M_{n-1}) \cdot p_{h-4} + x \cdot i(H_{n-1}) \cdot (p_{h-4} + p_{h-5} + p_{h-5} \cdot x), \\ i(K_n) &= i(M_{n-1}) \cdot p_{h-5} + x \cdot i(H_{n-1}) \cdot p_{h-6}.\end{aligned}$$

Substituting  $i(K_{n-1})$  and  $i(H_{n-1})$  to  $i(M_n)$  we obtain

$$i(M_n) = i(M_{n-1}) \cdot (p_{h-4} + x \cdot p_{h-5}) + i(H_{n-1}) \cdot (x \cdot (p_{h-4} + p_{h-5} + p_{h-5} \cdot x) + x^2 \cdot p_{h-6}) \quad (3)$$

Substituting  $i(H_{n-1})$  to obtained expression gives

$$\begin{aligned}i(M_n) &= i(M_{n-1}) \cdot (p_{h-4} + x \cdot p_{h-5}) + \\ &+ i(M_{n-2}) \cdot p_{h-4} \cdot (x \cdot (p_{h-4} + p_{h-5} + p_{h-5} \cdot x) + x^2 \cdot p_{h-6}) + \\ &+ x \cdot i(H_{n-2}) \cdot (p_{h-4} + p_{h-5} + p_{h-5} \cdot x) \cdot (x \cdot (p_{h-4} + p_{h-5} + p_{h-5} \cdot x) + x^2 \cdot p_{h-6}).\end{aligned}$$

Substituting  $i(H_{n-2})$  from (3) to this expression proves the claim of the theorem. ■

By setting  $x = 1$  to the recurrence relation from Theorem 6 we can obtain the recurrence relation for  $\Psi(M_n)$  in which coefficients would be total number of independent sets in different paths. Therefore, for a specific  $n$  and  $h$  we could calculate exact  $\Psi(M_n)$  from that recurrence relation. Now, we proceed to maximum independent set. We will establish size and number of such sets for meta-chains, i.e. independence number  $\alpha(M_n)$  and number of maximum independent sets  $\Psi_{\alpha(M_n)}(M_n)$ .



**Theorem 7.** *The independence number of  $M_n$ , for  $n \geq 1$ , is*

$$\alpha(M_n) = n \cdot \left\lfloor \frac{h}{2} \right\rfloor.$$

**Proof.** Let us consider set

$$S = \left\{ v_{2k-1}^{(j)} : 1 \leq j \leq n, 1 \leq k \leq \left\lfloor \frac{h}{2} \right\rfloor \right\}.$$

This set is obviously independent on  $M_n$  and each cycle  $C^{(j)}$  contains exactly  $\left\lfloor \frac{h}{2} \right\rfloor$  vertices from  $S$ . Since cycle  $C_h$  can contain at most  $\left\lfloor \frac{h}{2} \right\rfloor$  independent vertices, it follows that  $S$  is maximum independent set. ■

**Theorem 8.** *The number of maximum independent sets in  $M_n$ , for  $n \geq 2$ , is*

$$\Psi_{\alpha(M_n)}(M_n) = \begin{cases} 1, & \text{for } h \text{ even,} \\ \left(\frac{h-1}{2}\right)^{n-2} \cdot \left(\frac{h+1}{2}\right)^2, & \text{for } h \text{ odd.} \end{cases}$$

**Proof.** Because of Theorem 7, maximum independent set  $S$  in  $M_n$  must contain  $\left\lfloor \frac{h}{2} \right\rfloor$  on each cycle. Note that cycle  $C_h$  contains at most  $\left\lfloor \frac{h}{2} \right\rfloor$  independent vertices. If  $S$  contained cut vertex, it would be counted in  $\left\lfloor \frac{h}{2} \right\rfloor$  independent vertices on two different cycles, and consequently  $S$  wouldn't be maximum independent set. Therefore, maximum independent set  $S$  on  $M_n$  cannot contain cut vertices. Now note the following: if  $S$  is maximum independent set on  $M_n$  then  $v_1^{(j)} \in S$  for  $j = 2, \dots, h-1$ . We conclude

$$\begin{aligned} \Psi_{\alpha(M_n)} &= \Psi_{\left\lfloor \frac{h}{2} \right\rfloor - 1}(P_{h-3})^{n-2} \cdot \Psi_{\left\lfloor \frac{h}{2} \right\rfloor}(P_{h-1})^2 = \\ &= \Psi_{\alpha(P_{h-3})}(P_{h-3})^{n-2} \cdot \Psi_{\alpha(P_{h-1})}(P_{h-1})^2. \end{aligned}$$

The claim now follows from the number of maximum independent sets on path. ■

### *Extremality*

Now we want to establish extremal chain  $h$ -cacti with respect to total number of independent sets. For that purpose we define relation  $\preceq$  on polynomials. Let  $f(x) = \sum_{i=0}^n a_i x^i$  and  $g(x) = \sum_{i=0}^n b_i x^i$  be two polynomials. We say that  $f \preceq g$  if  $a_i \leq b_i$  for every  $i = 0, \dots, n$ . We say that  $f \prec g$  if  $f \preceq g$  and  $f \neq g$ . Now, we need following two lemmas.

**Lemma 9.** *Let  $A_n$  be a chain cactus of length  $n \geq 2$ . For  $2 \leq k \leq \lfloor \frac{h_n}{2} \rfloor$  holds*

$$i(A_n - v_1^{(n)}) \prec i(A_n - v_k^{(n)}).$$

**Proof.** Throughout the proof we will focus on  $C^{(n)}$ , so we will use notation  $C = C^{(n)}$ ,  $h = h_n$  and  $v_i = v_i^{(n)}$ . Also, we will denote  $G_k = A_n - v_k^{(n)}$ . The claim of the lemma is now

$$i(G_1) \prec i(G_k).$$

To prove it we need to prove

$$\begin{aligned} i(G_1) &\preceq i(G_k), \\ i(G_1) &\neq i(G_k). \end{aligned}$$

To prove

$$i(G_1) \preceq i(G_k),$$

it is sufficient to prove that for every independent set  $S$  on  $G_1$  there is corresponding (1) independent set  $S'$  on  $G_k$  which is (2) of the same size as  $S$  such that (3) mapping  $S \mapsto S'$  is injection.

CASE I:  $v_k \notin S$ . Then we define  $S' = S$ . Obviously,  $S'$  is well defined independent set on  $G_k$  of the same size as  $S$ , and this mapping is injection. Note that in this case  $v_1 \notin S'$  (since  $v_1 \notin S$ ).

CASE II:  $v_k \in S$ . Note that in this case  $v_{k+1} \notin S$ , and also  $v_1 \notin S$  (since  $v_1 \notin G_1$ ). We define  $S'$  in the following manner:

$$\begin{aligned} v \in S' &\iff v \in S && \text{for } v \in V \setminus V(C), \\ v_i \in S' &\iff v_{k+1-i} \in S && \text{for } 1 \leq i \leq k-1, \\ v_i \in S' &\iff v_{i+1} \in S && \text{for } k+1 \leq i \leq h-1. \end{aligned}$$

This construction is illustrated on Figure 3. First note that  $S'$  is well defined set of vertices from  $G_k$  since  $v_k \notin S'$  by definition. Now, note that  $S'$  is independent on  $G_k - v_h$  since  $S$  is independent on  $G_1$ . Since  $v_h \notin S'$  by construction, it follows that  $S'$  is independent on  $G_k$  too. Furthermore,  $S$  and  $S'$  are of the same size since their cardinalities obviously coincide on  $V \setminus V(C)$  and  $\{v_1, \dots, v_k\}$ . Also, because of  $v_{k+1} \notin S$  their cardinalities coincide on  $\{v_{k+1}, \dots, v_h\}$  too. Let us now show that  $S \mapsto S'$  is injection. If sets  $S$  differ on  $V \setminus V(C)$  or  $\{v_2, \dots, v_k, v_{k+2}, \dots, v_h\}$  then corresponding sets  $S'$  differ on  $V \setminus V(C)$  or  $\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_{h-1}\}$  respectively. Also, sets

$S$  can't differ on  $v_{k+1}$  since in this case  $v_{k+1} \notin S$ . Hence  $S \mapsto S'$  is injection and we have proved the claim for this case. Note that in this case  $v_1 \in S'$  (since  $v_k \in S$ ).

We still have to prove overall injectivity (across the cases) of mapping  $S \mapsto S'$ . Let now  $S_1$  be independent set from first case ( $v_k \notin S_1$ ) and  $S_2$  be independent set from second case ( $v_k \in S_2$ ). What remains to be proved is that  $S'_1 \neq S'_2$ . But we have noted that in the first case  $v_1 \notin S'_1$ , and in the second case  $v_1 \in S'_2$ . Hence,  $S'_1 \neq S'_2$  and we conclude that mapping  $S \mapsto S'$  is overall injection from independent sets on  $G_1$  to independent sets on  $G_k$  of the same size.

To prove that

$$i(G_1) \neq i(G_k)$$

it enough to find independent set  $S'$  on  $G_k$  for which there is no independent set  $S$  on  $G_1$  such that  $S \mapsto S'$ . Let  $v_j^{(n-1)}$  be the cut vertex on  $C^{(n-1)}$  of  $A_n$  such that  $v_j^{(n-1)} = v_h^{(n)}$ . Let us consider set  $S' = \{v_{j+1}^{(n-1)}, v_1^{(n)}, v_{h-1}^{(n)}\}$ . We claim that  $S'$  is such set. Suppose contrary, i.e. that there is independent set  $S$  on  $G_1$  such that  $S \mapsto S'$ . Then  $S$  should be from second case since  $v_1^{(n)} \in S'$ . From construction of second case follows that  $S = \{v_{j+1}^{(n-1)}, v_h^{(n)} = v_j^{(n-1)}, v_k^{(n)}\}$ . But such  $S$  is not independent because of edge  $e = v_{j+1}^{(n-1)}v_j^{(n-1)}$ . Therefore, we have contradiction. ■

Figure 3: With the proof of Lemma 9. Squared filled vertices are certainly included in independent set, while squared empty vertices are certainly excluded.

**Lemma 10.** *Let  $A_n$  be a chain cactus of length  $n \geq 2$ . For  $3 \leq k \leq \lfloor \frac{h_n}{2} \rfloor$  holds*

$$i(A_n - v_k^{(n)}) \prec i(A_n - v_2^{(n)}).$$

**Proof.** Again, throughout the proof we will focus on  $C^{(n)}$ , so we will use notation  $C = C^{(n)}$ ,  $h = h_n$  and  $v_i = v_i^{(n)}$ . Also, we will denote  $G_k = A_n - v_k^{(n)}$ . The claim of the lemma is now

$$i(G_k) \prec i(G_2).$$

To prove it we need to prove

$$\begin{aligned} i(G_k) &\preceq i(G_2), \\ i(G_k) &\neq i(G_2). \end{aligned}$$

To prove

$$i(G_k) \preceq i(G_2),$$

it is sufficient to prove that for every independent set  $S$  on  $G_k$  there is corresponding (1) independent set  $S'$  on  $G_2$  which is (2) of the same size as  $S$  such that (3) mapping  $S \mapsto S'$  is injection. Let  $S$  be independent set on  $G_k$ . We distinguish three cases.

CASE I:  $v_2 \notin S$ . Then we define  $S' = S$ . Obviously,  $S'$  is well defined independent set on  $G_2$  ( $v_2 \notin S'$  since  $v_2 \notin S$ ),  $S$  and  $S'$  are of the same size and  $S \mapsto S'$  is injection. Therefore, the claim is proved in this case. Note that in this case  $v_k \notin S'$  (since  $v_k \notin S$ ).

CASE II:  $v_2 \in S$  and  $v_{k+1} \notin S$ . Note that in this case  $v_1 \notin S$  and also  $v_k \notin S$  (since  $v_k \notin G_k$ ). We define  $S'$  in the following manner:

$$\begin{aligned} v \in S' &\iff v \in S && \text{for } v \in V \setminus V(C), \\ v_i \in S' &\iff v_{k+2-i} \in S && \text{for } 3 \leq i \leq k, \\ v_i \in S' &\iff v_i \in S && \text{for } k+1 \leq i \leq h. \end{aligned}$$

This construction is illustrated on Figure 4. First, note that  $S'$  is well defined set of vertices from  $G_2$  since  $v_2 \notin S'$  by definition. Further, if we denote  $e = v_k v_{k+1}$  we can see that  $S'$  is independent  $G_2 - e$  since  $S$  is independent on  $G_k$ . Since  $v_{k+1} \notin S'$  by construction (follows from  $v_{k+1} \notin S$ ) we conclude that  $S'$  is independent on  $G_2$  too. As for the size, sets  $S$  and  $S'$  are of the same size because for every vertex from  $S$  there is by definition corresponding vertex in  $S'$ . Further, if sets  $S$  differ on  $V \setminus V(C) \cup \{v_{k+1}, \dots, v_h\}$  or  $\{v_2, \dots, v_{k-1}\}$

then corresponding sets  $S'$  differ on  $V \setminus V(C) \cup \{v_{k+1}, \dots, v_h\}$  or  $\{v_3, \dots, v_k\}$  respectively. Also, sets  $S$  can't differ on  $v_1$  since  $v_1 \notin S$ . We conclude that mapping  $S \rightarrow S'$  of sets from this case is injection. Hence, we have proved the claim in this case. Note that in this case  $v_1 \notin S'$  and  $v_k \in S'$ .

CASE III.  $v_2 \in S$  and  $v_{k+1} \in S$ . We define  $S'$  in the following manner:

$$\begin{aligned} v \in S' &\iff v \in S && \text{for } v \in V \setminus V(C), \\ v_i \in S' &\iff v_{k+2-i} \in S && \text{for } 3 \leq i \leq k, \\ v_i \in S' &\iff v_{h+k+2-i} \in S && \text{for } k+2 \leq i \leq h, \\ v_1 &\in S'. \end{aligned}$$

This construction is illustrated on Figure 5. First note that  $S'$  is well defined set of vertices from  $G_2$  since  $v_2 \notin S'$  by definition. Now, let  $e = v_k v_{k+1}$ . Set  $S'$  is independent set on  $G_2 - e - v_h$  since  $S$  is independent on  $G_k$ . Edge  $e$  does not cause problems with independence of  $S'$ , since  $v_{k+1} \notin S'$  by definition. Also, no edge incident with  $v_h$  is problem since  $v_h \notin S'$  (since  $v_{k+2} \notin S$ , which is since  $v_{k+1} \in S$ ). Therefore,  $S'$  is independent on  $G_2$  too. As for the size, sets  $S$  and  $S'$  are of the same size because for every vertex from  $S$  there is by definition corresponding vertex in  $S'$ . Furthermore, if sets  $S$  differ on  $V \setminus V(C)$  or  $\{v_2, \dots, v_{k-1}, v_{k+2}, \dots, v_h\}$  then sets  $S'$  differ on  $V \setminus V(C)$  or  $\{v_3, \dots, v_k, v_{k+2}, \dots, v_h\}$  respectively. Sets  $S$  cannot differ on  $v_{k+1}$  or  $v_1$  since  $v_{k+1} \in S$  and  $v_1 \notin S$  for all  $S$  in this case. Therefore, mapping  $S \rightarrow S'$  of sets from this case is injection. Hence, we have proved the claim in this case too. Note that in this case  $v_1 \in S'$  and  $v_k \in S'$ .

What remains to be proved is that mapping  $S \rightarrow S'$  is overall injection (across the cases). Let  $S_1$  be independent set from first case,  $S_2$  from second case and  $S_3$  from third case. Then  $S'_1 \neq S'_2$  and  $S'_1 \neq S'_3$  since  $v_k \notin S'_1$  and  $v_k \in S'_2, S'_3$ . Also,  $S'_2 \neq S'_3$  since  $v_1 \notin S'_2$  and  $v_1 \in S'_3$ . Therefore, mapping  $S \rightarrow S'$  is overall injection.

To prove that

$$i(G_k) \neq i(G_k)$$

it enough to find independent set  $S'$  on  $G_2$  for which there is no independent set  $S$  on  $G_k$  such that  $S \mapsto S'$ . Let  $v_j^{(n-1)}$  be the cut vertex on  $C^{(n-1)}$  of  $A_n$  such that  $v_j^{(n-1)} = v_h^{(n)}$ . Let us consider set  $S' = \{v_{j+1}^{(n-1)}, v_1^{(n)}, v_k^{(n)}, v_{k+2}^{(n)}\}$ . We claim that  $S'$  is such set. Suppose contrary, i.e. that there is independent set  $S$  on  $G_1$  such that  $S \mapsto S'$ . Then  $S$  should be from third case since  $v_k^{(n)} \in S'$  and  $v_1^{(n)} \in S'$ . From construction of the third case follows that

$S = \left\{ v_{j+1}^{(n)}, v_h^{(n)} = v_j^{(n-1)}, v_2^{(n)}, v_{k+1}^{(n)} \right\}$ . But such  $S$  is not independent because of edge  $e = v_{j+1}^{(n-1)} v_j^{(n-1)}$ . Therefore, we have contradiction. ■

Figure 4: With the proof of Lemma 10. Squared filled vertices are certainly included in independent set, while squared empty vertices are certainly excluded.

Figure 5: With the proof of Lemma 10. Squared filled vertices are certainly included in independent set, while squared empty vertices are certainly excluded.

Setting  $x = 1$  in polynomials from Lemmas 9 and 10 we obtain following corollary.

**Corollary 11.** *Let  $A_n$  be a chain cactus of length  $n \geq 2$ . For  $3 \leq k \leq \lfloor \frac{h_n}{2} \rfloor$  holds*

$$\Psi(A_n - v_1^{(n)}) < \Psi(A_n - v_k^{(n)}) < \Psi(A_n - v_2^{(n)}).$$

Now we can proceed with the main theorem.

**Theorem 12.** Let  $A_n$  be a chain  $h$ -cactus of length  $n \geq 3$  such that  $A_n \neq M_n$  and  $A_n \neq O_n$ . Then

$$\Psi(O_n) < \Psi(A_n) < \Psi(M_n).$$

**Proof.** Let  $A_n$  be any chain  $h$ -cacti of length  $n \geq 3$  such that  $A_n \neq M_n$  and  $A_n \neq O_n$ . Then

$$\begin{aligned} i(A_n) &= x \cdot i(A_{n-1} - N[v_k^{(n-1)}]) \cdot i(P_{h-3}) + i(A_{n-1} - v_k^{(n-1)}) \cdot i(P_{h-1}) = \\ &= x \cdot i(A_{n-1} - N[v_k^{(n-1)}]) \cdot i(P_{h-3}) + i(A_{n-1} - v_k^{(n-1)}) \cdot (x \cdot i(P_{h-3}) + i(P_{h-2})) = \\ &= x \cdot i(P_{h-3}) \cdot \left( i(A_{n-1} - N[v_k^{(n-1)}]) + i(A_{n-1} - v_k^{(n-1)}) \right) + i(A_{n-1} - v_k^{(n-1)}) \cdot i(P_{h-2}). \end{aligned}$$

Setting  $x = 1$  in these polynomials we obtain

$$\Psi(A_n) = \Psi(P_{h-3}) \cdot \Psi(A_{n-1}) + \Psi(A_{n-1} - v_k^{(n-1)}) \cdot \Psi(P_{h-2}). \quad (4)$$

Note that the same holds for  $O_n$  and  $M_n$ . Now we will prove by induction on  $n$  the following three claims simultaneously

$$\begin{aligned} \Psi(O_{n-1} - v_1^{(n-1)}) &\leq \Psi(A_{n-1} - v_1^{(n-1)}), \\ \Psi(A_{n-1} - v_2^{(n-1)}) &\leq \Psi(M_{n-1} - v_2^{(n-1)}), \\ \Psi(O_n) &< \Psi(A_n) < \Psi(M_n). \end{aligned}$$

For  $n = 3$ , the first two claims follow from  $A_2 = O_2 = M_2$  and the third claim follows from (4), the fact that  $A_2 = O_2 = M_2$  and Corollary 11.

For  $n > 3$ , let us suppose that  $C^{(n-2)}$  of  $A_n$  is in  $j$ -position. Then we have

$$\begin{aligned} \Psi(A_{n-1} - v_1^{(n-1)}) &= \Psi(A_{n-2}) \cdot \Psi(P_{h-3}) + \Psi(A_{n-2} - v_j^{(n-2)}) \cdot \Psi(P_{h-4}). \\ \Psi(O_{n-1} - v_1^{(n-1)}) &= \Psi(O_{n-2}) \cdot \Psi(P_{h-3}) + \Psi(O_{n-2} - v_1^{(n-2)}) \cdot \Psi(P_{h-4}). \end{aligned}$$

Since

$$\Psi(A_{n-2}) > \Psi(O_{n-2})$$

by induction assumption, and also

$$\Psi(A_{n-2} - v_j^{(n-2)}) > \Psi(A_{n-2} - v_1^{(n-2)}) \geq \Psi(O_{n-2} - v_1^{(n-2)})$$

by Corollary 11 and induction assumption respectively, we obtain

$$\Psi(O_{n-1} - v_1^{(n-1)}) < \Psi(A_{n-1} - v_1^{(n-1)}). \quad (5)$$

In a similar fashion we have

$$\begin{aligned}\Psi(A_{n-1} - v_2^{(n-1)}) &= \Psi(A_{n-2}) \cdot \Psi(P_{h-4}) + \Psi(A_{n-2} - v_j^{(n-2)}) \cdot (\Psi(P_{h-4}) + 2\Psi(P_{h-5})), \\ \Psi(M_{n-1} - v_2^{(n-1)}) &= \Psi(M_{n-2}) \cdot \Psi(P_{h-4}) + \Psi(M_{n-2} - v_2^{(n-2)}) \cdot (\Psi(P_{h-4}) + 2\Psi(P_{h-5})).\end{aligned}$$

Since

$$\Psi(A_{n-2}) < \Psi(M_{n-2})$$

by induction assumption and

$$\Psi(A_{n-2} - v_j^{(n-2)}) < \Psi(A_{n-2} - v_2^{(n-2)}) \leq \Psi(M_{n-2} - v_2^{(n-2)})$$

by Corollary 11 and induction assumption respectively, we obtain

$$\Psi(A_{n-1} - v_2^{(n-1)}) < \Psi(M_{n-1} - v_2^{(n-1)}). \quad (6)$$

The inequality

$$\Psi(O_n) < \Psi(A_n) < \Psi(M_n)$$

now follows from (4) since  $\Psi(O_n) < \Psi(A_n) < \Psi(M_n)$  by induction assumption and since inequalities (5) and (6) hold. ■

Note that Lemmas 9 and 10 (and consequently Corollary 11) hold for general chain cacti. Therefore, theorem for general chain cacti, analogous to Theorem 12, can be proved. The only condition is that number of vertices in  $i$ -th cycle (for  $i = 1, \dots, n$ ) must be the same for  $A_n$ ,  $O_n$  and  $M_n$  of general case.

**Theorem 13.** *Let  $A_n$  be a chain cactus of length  $n$ , and let  $O_n$  and  $M_n$  be ortho- and meta- chain cacti of length  $n$  such that  $A_n$ ,  $O_n$  and  $M_n$  have the same number of vertices on cycle  $C^{(i)}$  for every  $i = 1, \dots, n$ . If  $A_n \neq O_n$  and  $A_n \neq M_n$ , then*

$$\Psi(O_n) < \Psi(A_n) < \Psi(M_n).$$

**Proof.** Analogous to that of Theorem 12. ■

For the end, we can propose some directions for further study. It would be interesting to establish in what relation tree cacti stand to chain cacti with respect to total number of independent sets.



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